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PIONEER PAPER*

A GENERALIZATION OF THE METHODS OF HEAT EXCHANGER ANALYSIS

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NOMENCLATURE

A, transfer area [ft^2];
B, function of m_{eo} and m_{et} defined under equation (5);
B', function of m_{et} defined under equation (5a);
B'', function of m_{eo} and m_{et} defined under equation (6);
c, unit heat capacity of liquid water [B.t.u./(lb °F)];
C, the capacity rate of the flow stream (see Table 1);
D, m_{eo} or $1/m_{eo}$ selected so as to be ≤ 1;
d, the symbol for differential;
e, base of natural logarithms;
G, superficial air mass velocity [lb dry air/hr ft^2 of superficial flow cross-section, $G = w/A_c$];
H, air absolute humidity [lb/lb dry air];
h, the unit film conductance for contact exchangers. For tube type exchangers, it may also include the effect of wall resistance (see Table 1);
I, air unit enthalpy, dry air component referred to 0°F datum, water component referred to liquid at 32°F datum [B.t.u./lb dry air];
L, liquid rate [lb/hr];
l, height of packing [ft];
m_e, slope of the equilibrium line;
m_o, slope of the operating line;
m_t, slope of the tie line;
q, exchanger transfer current (see Table 1);
s, unit heat capacity of air [B.t.u./(lb dry air °F)];
T, the potential in the stream. T is one of the stream intensive properties influenced by q (see Table 1);
t, temperature [°F];
U, the overall conductance (see Table 1);
V, the exchanger volume measured in the direction of one of the stream flows [ft^3];
w, air rate [lb/hr];
α, the transfer area per unit exchanger volume, dA/dV [ft^2/ft^3];
Δ, the potential difference causing the transfer current q (see Table 1);
∞, infinity;
$|x|$, the magnitude of the quantity x;
ln, logarithm to the base e.

Subscripts

a, stream a, also air stream;
b, stream b;
c, chains, also flow cross-section;
e, the equilibrium line;
h, hot granular material;
I, based on enthalpy potential;
i, the interface condition;
m, the log-mean of the terminal conditions [equation (4)];
o, the operating line, also 'overall';
t, the tie line;
cf, counterflow;
pf, parallel-flow;
ch, hot material to chains, also maximum chain temperature at any cross-section;
cc, minimum chain temperature at any cross-section;
ca, chains to air;
∞, operation of a counterflow exchanger of $NTU = \infty$;
1, 2, terminal conditions at entrance and exit of exchanger.

Dimensionless moduli

ε, the exchanger effectiveness, q/q_∞;
m_{eo}, the magnitude of the slope ratio equilibrium line/operating line, $|m_e/m_o|$;
m_{et}, the magnitude of the slope ratio equilibrium line/tie lines, $|m_e/m_t|$;
NTU, the number of exchanger transfer units defined by equation (3).§

I. BASIC IDEAS

A. *Introduction*

It is the objective of this report to:

(a) present methods of heat exchanger calculations

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§ In today's nomenclature N_{tu} is employed in place of NTU. This change was strongly urged by Professor Max Jacob.

in addition to the common method of the *log-mean* rate equation;

(b) present the *exchanger effectiveness* concept and demonstrate its use by application to several special problems;

(c) generalize the methods of calculation of counter-flow or parallel-flow heat exchangers so as to be applicable to analogous exchanger systems (cooling towers, absorption columns, and extraction columns).

Most of the concepts presented in this report have been available in the literature for some time [1, 2, 4, 5, 10, 11, 12]. However, these ideas are not commonly employed in either industry or teaching. Generalization and consolidation of these concepts may encourage their use.

Expressed in general symbolism, the following analysis applies to heat exchangers, mass transfer systems, or energy exchanger systems (involving a combination of heat and mass transfer). Table 1 defines the analogous variables and relates them to the nomenclature of the context.

The solution of the usual exchanger problem (energy or mass transfer) entails a combination of a suitable conservation relation with a rate equation, as is illustrated in the following development.

B. *Conservation relation*

In the two-stream exchanger system, a transfer current flows from one stream to the other. This current may consist of either energy or material or both. However, the usual problem is concerned with either one or the other. This transfer from one stream to the other influences some of the intensive properties [7] of the stream, as a result of an increase or decrease of content of the substance of interest transferred by the current. One of the influenced intensive properties is employed as a *potential* for current flow. It is possible then, by employing the *capacity rate* of the flow stream, to relate the potential change to the increase or decrease of storage. This relation is shown in applying

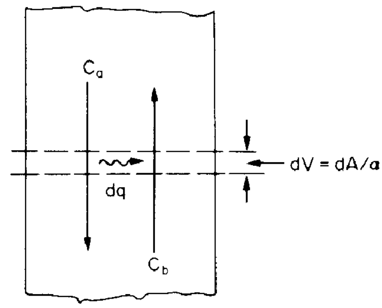

FIG. 1. Exchanger differential element.

the conservation principle (either energy or mass) to the steady-state, continuous-flow, counter-current exchanger (Fig. 1).

$$dq = C_a\,dT_a = C_b\,dT_b. \tag{1}$$

In this symbolism, dq is the differential transfer current, between streams a and b, occurring in the differential transfer section of volume dV and transfer area $dA = \alpha dV$. C denotes the capacity rate of the flow stream related to the potential for current flow, T (see Table 1).

The *operating line* is defined as the functional relation T_a vs T_b given by the conservation relations [equation (1)]. The slope of the operating line is

$$m_o = \frac{dT_a}{dT_b} = \frac{C_b}{C_a}. \tag{1a}$$

For a parallel-flow exchanger, equation (1) is modified by a sign change to

$$dq = -C_a\,dT_a = C_b\,dT_b. \tag{1b}$$

Then

$$m_o = \frac{dT_a}{dT_b} = -\frac{C_b}{C_a}. \tag{1c}$$

C. *Departure from equilibrium*

If at any section of the exchanger the two streams are

Table 1. The general symbolism applied to some particular systems

System	Current (q)	Capacity rate (C)	Current potential (T)	Unit conductance (U, h)
1. Heat exchangers	Thermal (B.t.u./hr)	Flow rate × unit heat capacity [B.t.u./(hr °F)]	Temperature (°F)	Heat transfer [B.t.u./(hr ft^2 °F)]
2. Cooling towers H_2O–air	Energy (B.t.u./hr)	Flow rate (lb/hr)	Gas phase: unit enthalpy (B.t.u./lb dry air). Liquid phase: temperature (°F)	Energy transfer gas phase: [B.t.u./(hr ft^2 B.t.u./lb)] liquid phase: [B.t.u./(hr ft^2 lb/lb)]
3. Adiabatic humidifiers H_2O–air	Mass (lb/hr)	Flow rate (lb/hr)	Gas phase: concentration (humidity, H) (lb/lb dry air)	Mass transfer [lb/(hr ft^2 lb/lb)]
4. Absorption and/or stripping columns	Mass (lb/hr, mol/hr)	Flow rate (lb/hr, mol/hr)	Concentration of diffusing material (lb/lb inert, mol/mol inert)	Mass transfer [mol/(hr ft^2 mol/mol)] [lb/(hr ft^2 lb/lb)]

in equilibrium, no transfer current will flow. The departure from equilibrium causes the transfer current, and the current is always in a direction tending towards equilibrium.

The *equilibrium line* is a plot of the co-existing equilibrium potentials of the substance of stream a vs the substance of stream b. This relation can only be determined by experiment and is known for a great number of systems. Since the Second Law of Thermodynamics provides the criterion for thermodynamic equilibrium, it follows that the equilibrium relation employed should be in conformity with this principle. For heat transfer, equilibrium obtains when $T_a = T_b$; thus, for this condition, the equilibrium line, T_a vs T_b in equilibrium with T_a, is a straight line of unity slope. In general, however, the equilibrium line is not straight. A typical configuration of operating and equilibrium lines is indicated in Fig. 2.

Two equilibrium conditions correspond to any particular operating condition (T_a, T_b); namely, T_{aeb} of stream a relative to stream b, and T_{bea} of stream b relative to stream a. $(T_a - T_{aeb}) = \Delta_a$ expresses the departure from equilibrium of stream a relative to stream b, based on the a substance potential; and similarly $(T_{bea} - T_b) = \Delta_b$ expresses the departure of stream b from equilibrium with stream a, based on the b substance potential.

The two streams in the exchanger system are separated by a phase interface in contact exchangers (cooling towers, absorption columns, etc.) or by a metal wall in tube-type heat exchangers. The potentials at the interface (or one face of the separating wall) are denoted by T_{ia} and T_{ib}, corresponding to either stream a and b, respectively. The usual practice is to assume that T_{ia} and T_{ib} are co-existing equilibrium magnitudes. This is a consequence of the basic assumption that the departure from equilibrium has no infinite gradients in the direction of current flow. In the case of heat exchangers, as temperature is the potential in both streams, the foregoing statement specifies a continuity of temperature variation in the direction of current flow. For cooling towers, absorption, stripping columns, and humidifiers, there is a discontinuity in the usually employed potentials at the interface, but

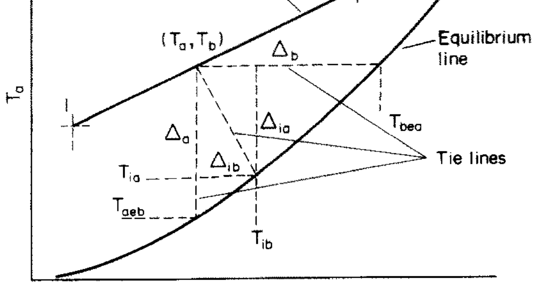

FIG. 2. Geometrical representation of the operating, equilibrium and tie lines.

the free energy variation is continuous. The potential differences $(T_a - T_{ia}) = \Delta_{ia}$ and $(T_{ib} - T_b) = \Delta_{ib}$ express the departure of streams a and b, respectively, from equilibrium with the interface (or wall) condition.

The straight lines joining the operating condition (T_a, T_b) to the equilibrium conditions on the equilibrium line (T_{aeb}, T_b), (T_a, T_{bea}), (T_{ia}, T_{ib}) are termed the *tie lines*.

D. *The rate equations*

The departure from equilibrium may be employed as a potential difference for current flow q, as expressed in the following rate equations:

$$\left.\begin{aligned} dq &= U_a(T_a - T_{aeb})dA = U_a \Delta_a dA \\ &= U_b(T_{bea} - T_b)dA = U_b \Delta_b dA \\ &= h_a(T_a - T_{ia})dA = h_a \Delta_{ia} dA \\ &= h_b(T_{ib} - T_b)dA = h_b \Delta_{ib} dA. \end{aligned}\right\} \quad (2)$$

In these expressions, U denotes the unit overall conductance, stream a to stream b, and h denotes the unit partial conductance, stream a or b to the interface condition. In the case of contact exchangers, h is the film conductance.

The slope of the tie line is established from these rate equations as

$$m_t = -\frac{\Delta_{ia}}{\Delta_{ib}} = -\frac{h_b}{h_a}. \quad (2a)$$

E. *The NTU expressions*

Combination of the suitable conservation expression, equation (1), with one of the rate equations (2), yields a solution of the usual exchanger problem, This combination produces the following forms:

$$\left.\begin{aligned} \int_1^2 \frac{dT_a}{\Delta_a} &= \int_1^2 \frac{U_a dA}{C_a} = NTU_a \\ \int_1^2 \frac{dT_b}{\Delta_b} &= \int_1^2 \frac{U_b dA}{C_b} = NTU_b \\ \int_1^2 \frac{dT_a}{\Delta_{ia}} &= \int_1^2 \frac{h_a dA}{C_a} = NTU_{ia} \\ \int_1^2 \frac{dT_b}{\Delta_{ib}} &= \int_1^2 \frac{h_b dA}{C_b} = NTU_{ib} \end{aligned}\right\} \quad (3)$$

The NTU denotes the Number of Exchanger Transfer Units, defined by either of the integrals integrated from one terminal condition to the other [4]. NTU_a and NTU_b are overall magnitudes, while NTU_{ia} and NTU_{ib} are effective film magnitudes for contact exchangers.

The integration of these expressions provides the solution to the following types of counterflow exchanger problems (see Fig. 3). Given the capacity rates C_a, C_b and the equilibrium line,

(a) determine NTU_a or NTU_b from the operating terminal potentials (T_{a1}, T_{b1}), (T_{a2}, T_{b2}) (Fig. 3a),

(b) determine the remaining terminal potentials from either NTU_a or NTU_b, and the terminal potentials at either 1 or 2, (Fig. 3b),

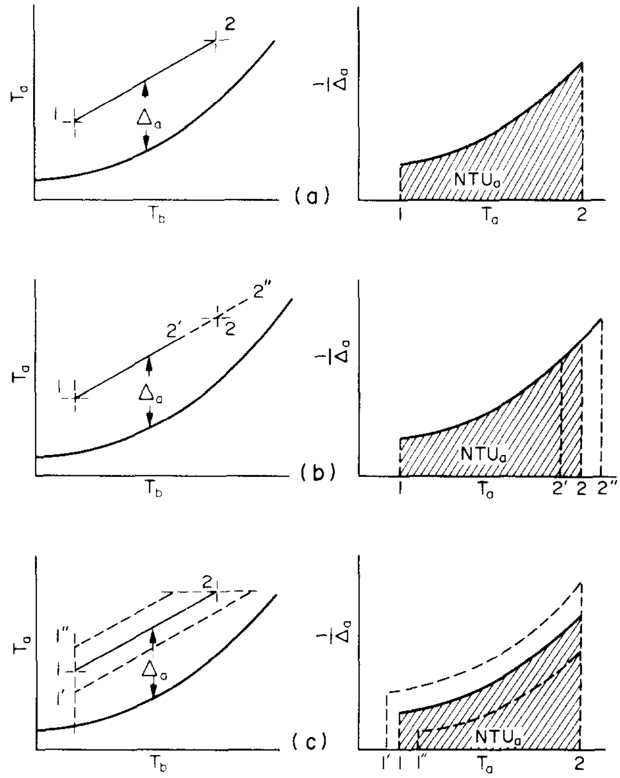

FIG. 3. Integration required to establish the *NTU*.

(c) determine the remaining terminal potentials either from NTU_a or NTU_b, and the terminal potentials of stream *a* at 1 and stream *b* at 2 (or vice versa) (Fig. 3c).

The operating line is generally straight because the capacity rates C_a, C_b are usually sensibly constant. The equilibrium line may not be straight: if not, the integration must be performed graphically or numerically. This can readily be accomplished by plotting $1/\Delta_a$ vs T_a or $1/\Delta_b$ vs T_b, obtained from the graph of the equilibrium and operating lines (Fig. 2), and then integrating the area under the resulting curve; this procedure is indicated in Fig. 3. For problem (a), evaluating the area under this curve between the known terminal conditions directly determines the *NTU*. For problem (b), a successive approximation solution of the unknown terminal conditions to yield an area equal to the known *NTU* establishes the desired terminal conditions. For problem (*c*), successive approximation selection of the position of the operating line (slope constant) so that the area under the curve of $1/\Delta$ vs T yields the given *NTU* will allow the determination of the proper operating line position, which establishes the unknown terminal conditions. These methods are indicated graphically in Figs. 3a, b, and c, respectively. The graphical representation is useful even though numerical methods of calculation may be employed, because it allows a clear visualization of the significance of the operations.

The foregoing analysis applies to systems where U is not a function of transfer area A (or U a constant with exchanger volume V for contact type exchangers, such as packed or spray systems). However, the graphical method may also be applied in the solution of problems where U is a function of T (and hence of A) by plotting $1/U\,\Delta$ vs T. Then the area under the curve is equal to

$$\int_1^2 \frac{\mathrm{d}T}{U\,\Delta} = \frac{A}{C}.$$

The performance of an exchanger system may be compared to the best possible performance to yield the *effectiveness of the exchanger* [2, 5, 9, 10, 11]. The effectiveness, ε, is defined as the ratio of the actual transfer current to the maximum possible current for the given flow rates and entering terminal conditions. The maximum possible transfer would be that obtained in a counterflow exchanger of infinite transfer area ($NTU = \infty$).

Figure 4 compares the operating line of the actual counterflow exchanger to that of an exchanger with $NTU = \infty$. For constant capacity rates, it is seen that

$$\varepsilon = \frac{q}{q_\infty} = \frac{(T_{a2} - T_{a1})}{(T_{a2} - T_{a1})_\infty} = \frac{(T_{b2} - T_{b1})}{(T_{b2} - T_{b1})_\infty}.$$

This ratio may be readily evaluated either graphically or by numerical methods from the entering terminal conditions T_{b1}, T_{a2}, and the equilibrium line data. For

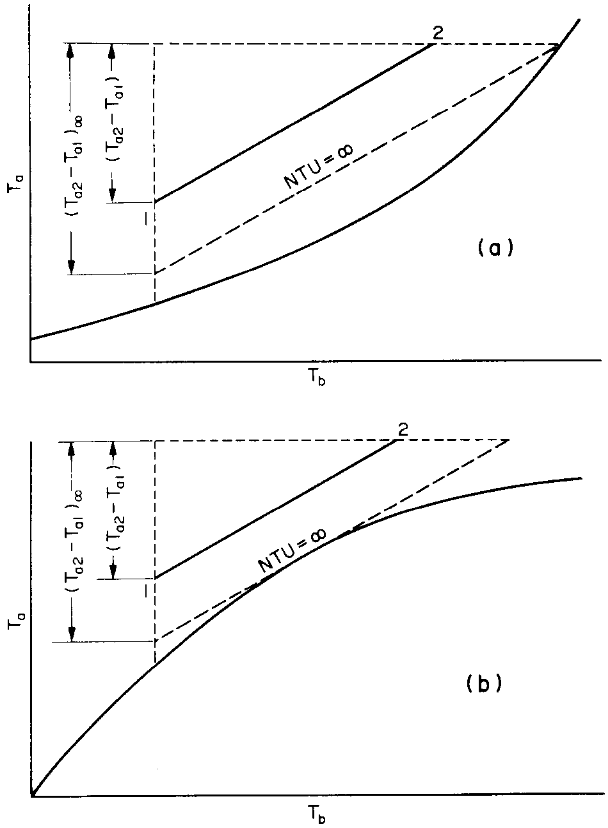

FIG. 4. Operating line for infinite NTU to establish q_∞ in the effectiveness definition.

a substance possessing an equilibrium line of the type indicated in Fig. 4b, the exchanger of $NTU = \infty$ (best possible exchanger) may not provide equilibrium of the two streams at entrance or exit, as pictured in Fig. 4a. If not, equilibrium will exist at some intermediate condition (T_{ax}, T_{bx}) where the operating line is tangent to the equilibrium line as in Fig. 4b. It is thermodynamically impossible for the operating line to cross the equilibrium line.

Since the counterflow exchanger with $NTU = \infty$ provides the maximum possible transfer, regardless of the nature of the actual exchanger (parallel, cross, counter, or mixed-flow), the effectiveness expresses the approach to perfection. Thus ε is useful as a design criterion when comparing the economics of exchanger area vs total current [2, 9]. It is also useful in extrapolating performance of a given exchanger to predict operation under new conditions.

When the equilibrium line is straight, or when it can be closely approximated by a straight line between terminal equilibrium conditions, an analytical integration of $\int_1^2 \mathrm{d}T/\Delta$ can be made to yield the following equivalent but differently stated solutions:

$$\left.\begin{aligned} \frac{T_{a2} - T_{a1}}{\Delta_{am}} &= NTU_a \\ \frac{T_{b2} - T_{b1}}{\Delta_{bm}} &= NTU_b \end{aligned}\right\} \qquad (4)$$

where

$$\frac{1}{\Delta_m} = \frac{1}{T_2 - T_1}\int_1^2 \frac{\mathrm{d}T}{\Delta} = \frac{\ln\left(\dfrac{\Delta_2}{\Delta_1}\right)}{\Delta_2 - \Delta_1} \qquad (4a)$$

and

$$\varepsilon_{cf} = \frac{1 - e^{-(NTU)B}}{1 - De^{-(NTU)B}} \qquad (5)$$

$D = m_{eo}$ or $1/m_{eo}$, selected to be ≤ 1,

$$B = \left|\frac{1 - m_{eo}}{1 + m_{et}}\right| \quad \text{when } NTU_{ia} \text{ is used}$$

and

$$B = \left|\frac{1 - \dfrac{1}{m_{eo}}}{1 + \dfrac{1}{m_{et}}}\right| \quad \text{when } NTU_{ib} \text{ is used,}$$

m_{eo} = magnitude of the slope ratio

$$\frac{\text{equilibrium line}}{\text{operating line}} = \left|\frac{m_e}{m_o}\right|$$

m_{et} = magnitude of the slope ratio

$$\frac{\text{equilibrium line}}{\text{tie line}} = \left|\frac{m_e}{m_t}\right|$$

$m_o = C_b/C_a$ and $|m_t| = h_b/h_a$.

On the basis of overall NTUs, B reduces to

$$B = |1 - m_{eo}| \quad \text{when } NTU_a \text{ is used}$$

and

$$B = \left|1 - \frac{1}{m_{eo}}\right| \quad \text{when } NTU_b \text{ is used.}$$

When $m_{eo} = 1$, the form of equation (5) becomes indeterminate: in this case the usual algebraic methods yield

$$\varepsilon_{cf} = \frac{1}{\left[1 + \frac{1}{(NTU)B'}\right]} \tag{5a}$$

$$B' = \frac{1}{[1 + m_{et}]} \quad \text{when } NTU_{ia} \text{ is used}$$

$$B' = \frac{1}{\left[1 + \frac{1}{m_{et}}\right]} \quad \text{when } NTU_{ib} \text{ is used.}$$

Equations (4) are a rearrangement of the log-mean rate equation

$$q = C(T_2 - T_1) = \Delta_m \int_1^2 U\,\mathrm{d}A.$$

This solution is strictly applicable only to systems involving straight equilibrium and operating lines and constant capacity rates C.

The forms indicated [equations (4)] also apply to the parallel-flow system. The effectiveness expression for the parallel-flow system is different from equation (5), namely

$$\varepsilon_{pf} = \frac{1 - e^{-(NTU)B''}}{1 + D} \tag{6}$$

$D = m_{eo}$ or $1/m_{eo}$, selected to be ≤ 1,

$$B'' = \frac{1 + m_{eo}}{1 + m_{et}} \quad \text{when } NTU_{ia} \text{ is used}$$

and

$$B'' = \frac{1 + \frac{1}{m_{eo}}}{1 + \frac{1}{m_{et}}} \quad \text{when } NTU_{ib} \text{ is used.}$$

If one of the capacity rates is very large compared to the other, $m_{eo} = \infty$ or 0, the counterflow and parallel-flow effectiveness expressions [equations (5) and (6)] yield the same result. This situation exists in condensers, evaporators, and steam heaters, and the reduced equations applicable to this case are of simple form.

For

$$C_a = \infty,\ C_b \neq 0,\ m_o = 0,$$

$$\varepsilon_{cf} = \varepsilon_{pf} = 1 - e^{-(NTU_{ib})B'} \tag{5b}$$

$$B' = \frac{1}{1 + \frac{1}{m_{et}}}.$$

For

$$C_a \neq 0,\ C_b = \infty,\ m_o = \infty,$$

$$\varepsilon_{cf} = \varepsilon_{pf} = 1 - e^{-(NTU_{ia})B'} \tag{5c}$$

$$B' = \frac{1}{1 + m_{et}}.$$

That is, NTU_{ia} cannot be employed when $C_a = \infty$ and similarly for NTU_{ib} when $C_b = \infty$.

The effectiveness expressions (5), (5a), and (6) are more useful than the orthodox log-mean rate equation for certain exchanger problems. For instance, if the NTU and some of the terminal conditions are specified and it is desired to evaluate the remaining terminal conditions, the effectiveness expression will by direct

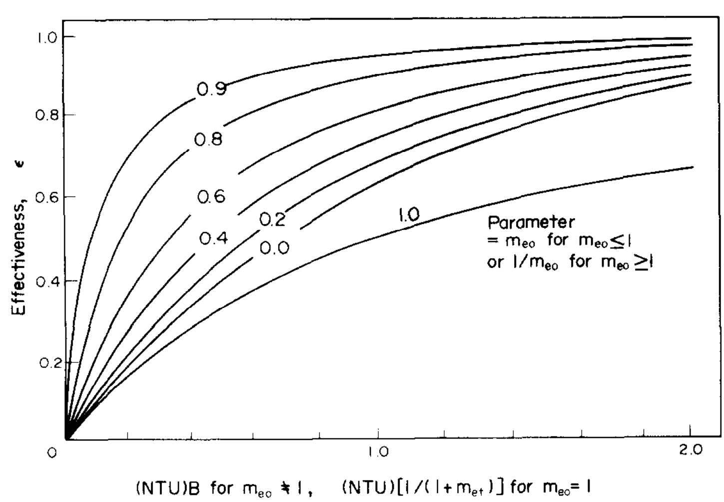

FIG. 5. Effectiveness NTU relationship for counterflow, equations (5) and (5a).

computation suffice; while if the log-mean rate equation were to be employed a successive approximation calculation would have to be used. In the design problem, however, where the terminal conditions are given and it is desired to evaluate the necessary NTU, the log-mean rate equation yields more direct results.

For those cases where capacity rates and conductances can be considered constant, and the equilibrium line is straight, exchanger calculations can be simplified by considering the dimensionless groupings indicated in equations (4), (5), and (6) as system properties to be treated as entities. These groupings are

$$\varepsilon = q/q_\infty = (T_2 - T_1)/(T_2 - T_1)_\infty \qquad (7)$$

the exchanger effectiveness

NTU = the number of transfer units

m_{eo} = the slope ratio magnitude, equilibrium line/operating line

m_{et} = the slope ratio magnitude, equilibrium line/tie line.

Figure 5 reveals the effectiveness expression, equation (5), in terms of the above dimensionless groupings.

The available published information on correction factors to be applied to the calculated log-mean potential difference in evaluating the true mean potential difference for cross-flow and mixed-flow exchangers with constant conductances is conveniently presented employing some of these ratios [3]. The R factor of the cited paper is the capacity rate ratio, while the P factor is the effectiveness ε for $C_a/C_b < 1$, or $\varepsilon/(C_a/C_b)$ for $C_a/C_b > 1$ (where in the nomenclature of the paper mentioned t corresponds to the potential in stream a and T is the potential in stream b).

II. SPECIAL PROBLEMS

The following problems demonstrate applications of the method of analysis considered.

A. *Design of a counterflow cooling tower*

The method of design of this unit illustrates the application of suitable rate and continuity equations (equations (1) and (2), and Table 1) when the equilibrium line is not straight.

A cooling tower is to cool 50,000 lb of water per hour from 110°F to 80°F by direct contact with an air stream in counterflow with the water. The air rate is to be 75,000 lb/h, with an initial state: temperature 75°F, humidity 0.0130 lb/lb.

The conductance for the particular type of packing to be employed has been experimentally determined as [8]:

$$h_l\alpha = 6.7\left(\frac{w}{A_c}\right)^{0.48} = 6.7G^{0.48}.$$

The indicated conductance applies to the gas side only. Intimate and frequent mixing of the liquid phase, provided by the particular packing arrangement, results in a relatively small liquid-side resistance and justifies the use of this unit conductance as an overall magnitude.

The operation of the tower is defined by the expression [equation (3)]:

$$\int_{I_1}^{I_2} \frac{dI}{I_i - I} = NTU = \frac{h_l\alpha l}{G}.$$

The desired tower volume is the product of the cross-section area, A_c, and the height, l, of the packed section. Since the mass velocity, w/A_c, must be specified so that h_l may be determined, A_c must first be chosen. This choice depends upon both dynamical and economic considerations [12], which here will be eliminated by the specification of a 10 ft/sec superficial air velocity. With air density of 0.727 lb/ft^3,

$$G = \frac{w}{A_c} = 2620 \text{ lb dry air per ft}^2 \text{ hr}.$$

Since $w = 75{,}000$ lb/hr, the cross-section will be

$$A_c = \frac{75{,}000}{2{,}620} = 28.6 \text{ ft}^2.$$

The overall conductance $h_l\alpha = 6.7(w/A_c)^{0.48}$ and the NTU expression becomes

$$NTU = \frac{6.7\left(\frac{w}{A_c}\right)^{0.48} l}{\left(\frac{w}{A_c}\right)} = 0.1115l.$$

The evaluation of the integral of equation (7), determining the NTU, may be accomplished graphically, as indicated in the following procedure, or numerically.

Figure 6 contains the equilibrium (saturation) line for an air–water vapor system and the operating line, specified as follows:

Operating line

Conditions at 2 (air inlet):

Water temperature = 80°F;

Air enthalpy, as determined from given state [6]

$$I = st + 1061H$$

$$I_2 = (0.245)75 + 1061(0.013) = 32.2 \text{ B.t.u./lb.}$$

Conditions at 1 (water inlet):

Water temperature = 110°F.

Air enthalpy, from an overall energy balance

$$w(I_1 - I_2) = Lc(T_1 - T_2),$$

$$I_1 = I_2 + (Lc/w)(T_2 - T_1) = 52.2 \text{ B.t.u./lb.}$$

The slope of the operating line is also defined by the equation of an energy balance on a differential length of the exchanger [equation (1a)]

$$Lc\,dT = w\,dI$$

$$\frac{dI}{dT} = \frac{Lc}{w} = \frac{C_b}{C_a} = \frac{(50{,}000)(1)}{(75{,}000)} = 0.667 \text{ B.t.u./(lb °F)}.$$

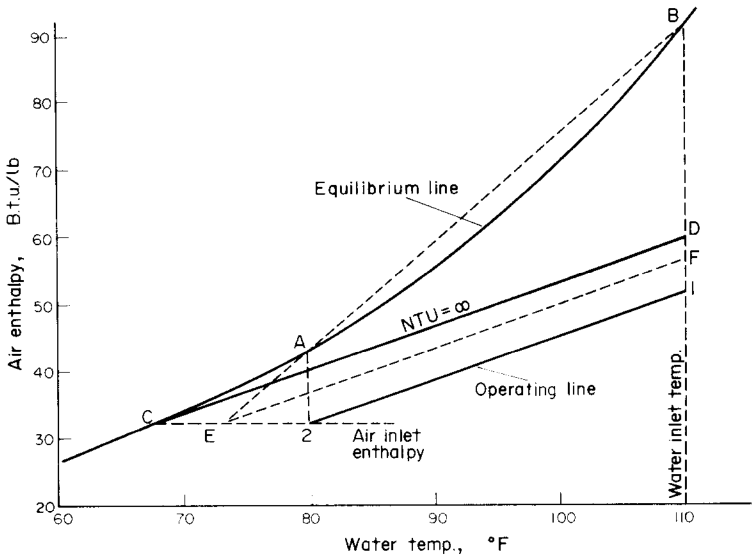

FIG. 6. Operating line, equilibrium line graph for the water cooling tower problem.

The *equilibrium line* is specified by the equation

$$I = (0.45H_i + 0.24)t_i + 1061H_i$$

where H_i is evaluated at the saturation temperature t_i [6].

Figure 6 yields information for the preparation of a curve of $1/(I_i - I)$ as a function of I: Fig. 7 and the area under this curve, 0.995 (dimensionless), evaluates

$$\int_{I_1}^{I_2} \frac{dI}{I_i - I}$$

the NTU required to accomplish the desired performance

$$0.995 = NTU = 0.1115l$$

and $l = 8.92$ ft, the packing height required with $A_c = 28.6\ \text{ft}^2$ as the packing cross-section.

A rough approximation to the tower size could have been more readily obtained by use of the log-mean of the terminal potentials

$$\Delta_m = \frac{\Delta_1 - \Delta_2}{\ln \dfrac{\Delta_1}{\Delta_2}} = \frac{39.9 - 11.5}{\ln \dfrac{39.9}{11.5}} = 23.0\ \text{B.t.u./lb.}$$

From equation (4),

$$\frac{I_1 - I_2}{\Delta_m} = NTU = \frac{52.2 - 32.3}{23.0} = 0.870$$

$$NTU = 0.1115l = 0.870$$

$$l = 7.81\ \text{ft.}$$

The error in the calculated height of the packing section due to the use of the log-mean potential

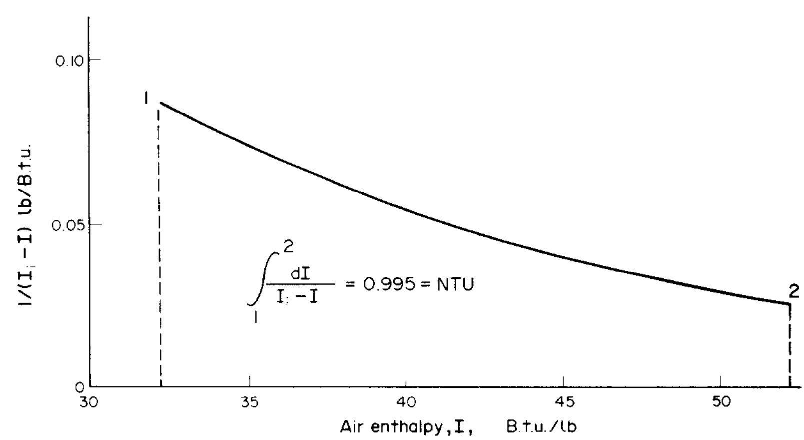

FIG. 7. NTU integration for the cooling tower problem.

(assumption of a straight equilibrium line AB, Fig. 6) is

$$\frac{8.92 - 7.81}{8.92} = 12.5\%.$$

The difference between the correct NTU, as evaluated graphically, and that obtained by the assumption of a log-mean potential difference is here considerable, because of the appreciable deviation of the equilibrium line from a straight line, AB, between the terminal points. When the cooling range is smaller, this deviation is less and the NTU as evaluated by the log-mean potential difference more closely approximates the correct NTU as defined by equation (7).

The effectiveness of the tower is the ratio of the actual energy transfer rate to the maximum possible transfer rate. This maximum is indicated by the operating line CD, which defines the ultimate in performance, as attainable by a tower having an $NTU = \infty$. From Fig. 6,

$$\varepsilon = \frac{T_1 - T_2}{T_1 - T_C} = \frac{110 - 80}{110 - 67.8} = 71\%.$$

Because of the large change in slope of the equilibrium line, the use of the analytical expression for effectiveness (based on a straight equilibrium line) will involve an error similar to that incurred in the NTU calculated by the use of the log-mean potential difference. From the lines on Fig. 6, or from equation (5) with $NTU = 0.870$, this erroneous effectiveness is

$$\varepsilon = \frac{T_1 - T_2}{T_1 - T_E} = 81\%.$$

When the tower NTU is known and the tower performance is to be evaluated, an initial approximation of the effectiveness by equation (5) will facilitate the necessary successive approximation graphical solution previously described.

B. *The chain type regenerative heat exchanger*

The utilization of the effectiveness concept in the analysis of a particular type of regenerative heat exchanger illustrates the application of this method of analysis to a system more complex than that of the preceding example.

The system to be considered is an exchanger for providing heat transfer from a stream of hot granular material to a cold air stream. The hot material passes along the bottom of a rotating cylindrical drum. Chains attached to the drum experience periodic submergence in the hot material and then exposure to the cold air stream. The flow of the hot granular material is countercurrent to the air flow. A section of the exchanger is illustrated in Fig. 8.

The following system idealizations are required for this analysis:

(1) the steel chains are of infinite radial and zero longitudinal conductivity, and all heat transfer takes place via these chains;

(2) the hot material is sufficiently mixed so that its temperature is uniform at any cross-section of the unit, the same condition applies to the cold air stream;

(3) steady state prevails with respect to all material flow rates, hot solid, cold air, and chains;

(4) the resistances to heat flow from hot material to chains, and from chains to cold air, are independent of circumferential and longitudinal locations;

(5) the chain capacity rate per unit exchanger length is constant; and

(6) the exposed area of the chains, per unit exchanger length, is constant.

The diagrams of Fig. 9 reveal the character of the temperature conditions presumed by the above idealizations.

An energy balance on a longitudinal section, dx, of the exchanger yields:

$$dq_{ch} = C_a\,dt_a = C_h\,dt_h. \quad (8)$$

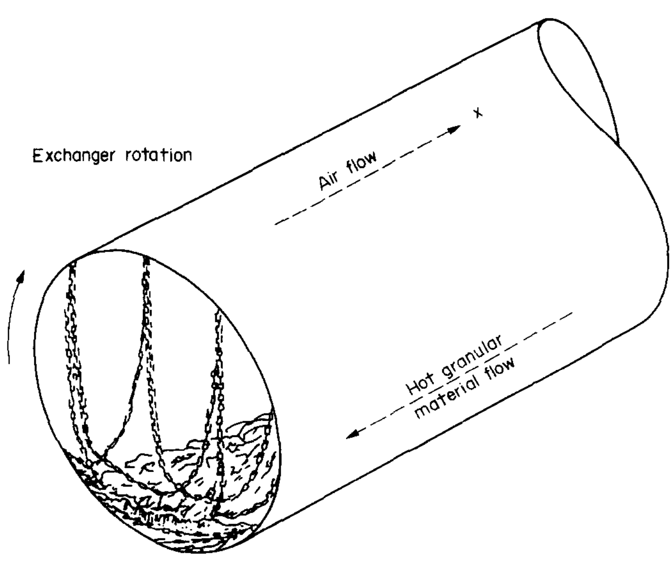

FIG. 8. Regenerative heat exchanger, chain type, for air cooling of a granular material.

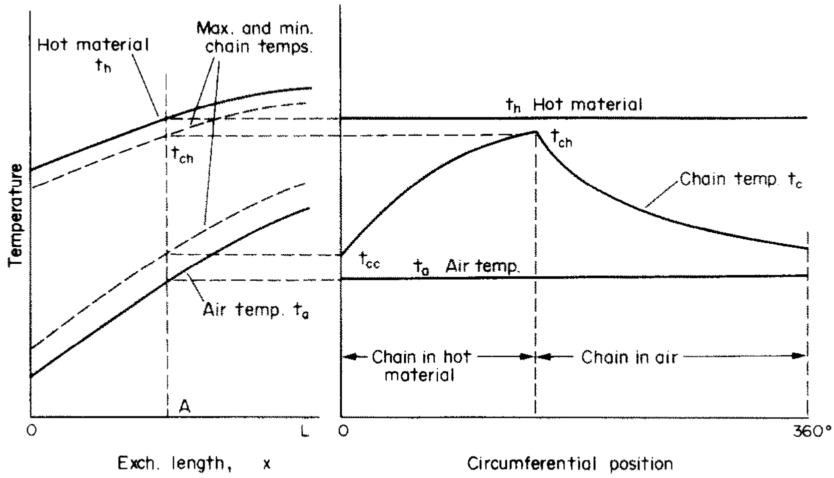

FIG. 9. Relative temperature conditions for the heat exchanger of Fig. 8.

From Fig. 9 the effectiveness of the heat transfer from the hot material to the chains, in the differential section, dx, is defined as

$$\varepsilon_{ch} = \frac{dq_{ch}}{dq_{ch\,max}} = \frac{dq_{ch}}{(t_h - t_{cc})dC_c}. \tag{9}$$

For the heat transfer from the chains to the air, the effectiveness is defined as

$$\varepsilon_{ca} = \frac{dq_{ca}}{dq_{ca\,max}} = \frac{dq_{ca}}{(t_{ch} - t_a)dC_c}. \tag{10}$$

The hot granular material and cold air temperatures are considered as uniform over the cross-section in equations (9) and (10) by virtue of idealization number (2). The analytical expressions of effectiveness pertaining to the heat transfers of equations (9) and (10) are the same as equation (5c). For the transfer from the hot material to the chains

$$\varepsilon_{ch} = 1 - e^{-NTU_{ch}}. \tag{9a}$$

For the transfer from the chains to the air

$$\varepsilon_{ca} = 1 - e^{-NTU_{ca}}. \tag{10a}$$

In these expressions the NTUs may be evaluated from

$$NTU_{ch} = \frac{U_{ch}A_{ch}}{C_c}$$

$$NTU_{ca} = \frac{U_{ca}A_{ca}}{C_c}$$

in accordance with the listed idealizations (3), (4), and (6).

An energy balance on the chains yields

$$dq_{ch} = (t_{ch} - t_{cc})dC_c. \tag{11}$$

Combination of equations (9), (10), and (11) to eliminate t_{cc} and t_{ch} yield

$$(t_h - t_a) = \frac{dq_{ch}}{dC_c}\left(\frac{1}{\varepsilon_{ch}} + \frac{1}{\varepsilon_{ca}} - 1\right). \tag{12}$$

The energy balance equation (8) can be arranged as

$$dt_h - dt_a = \frac{dq_{ch}}{C_h} - \frac{dq_{ch}}{C_a} = d(t_h - t_a)$$

$$= dq_{ch}\left(\frac{1}{C_h} - \frac{1}{C_a}\right) \tag{13}$$

dq_{ch} is eliminated by combining equations (12) and (13) to obtain

$$t_h - t_a = \frac{d(t_h - t_a)}{dC_c}\left(\frac{1}{\dfrac{1}{C_h} - \dfrac{1}{C_a}}\right)\left(\frac{1}{\varepsilon_{ca}} + \frac{1}{\varepsilon_{ch}} - 1\right). \tag{14}$$

Separating variables and integrating,

$$\int_1^2 \frac{d(t_h - t_a)}{(t_h - t_a)} = \int_0^{C_c}\left(\frac{1}{C_h} - \frac{1}{C_a}\right)\left(\frac{1}{\dfrac{1}{\varepsilon_{ca}} + \dfrac{1}{\varepsilon_{ch}} - 1}\right)dC_c. \tag{15}$$

The left side of equation (15) defines the difference of the overall NTUs, NTU_h and NTU_a [equation (3)].

$$NTU_h - NTU_a = \left(\frac{1}{C_{min}} - \frac{1}{C_{max}}\right)\left(\frac{1}{\dfrac{1}{\varepsilon_{ch}} + \dfrac{1}{\varepsilon_{ca}} - 1}\right)C_c$$

$$= (AU)_0\left(\frac{1}{C_{min}} - \frac{1}{C_{max}}\right) \tag{16}$$

where $C_{min} = C_h$, $C_{max} = C_a$, and

$$(AU)_0 = \left(\frac{1}{\dfrac{1}{\varepsilon_{ch}} + \dfrac{1}{\varepsilon_{ca}} - 1}\right)C_c \tag{16a}$$

$(AU)_0$ may be considered as the effective overall conductance-area product. It can be used in the log-mean rate equations (4). Also, from equation (16)

$$NTU_h - NTU_a = \frac{(AU)_0}{C_{min}}\left(1 - \frac{C_{min}}{C_{max}}\right)$$
$$= (NTU)B \qquad (17)$$

which, together with the ratio of the capacity rates, C_{min}/C_{max}, will yield the exchanger effectiveness, when substituted into equation (5).

The foregoing analysis may be used as the basis of design or performance calculations. Its application to the following data, typical of such exchangers, illustrates the prediction of the exchanger performance. The configuration of the exchanger is similar to that indicated in Fig. 8.

From equation (17),

$$(NTU)B = \frac{(AU)_0}{C_{min}}\left[1 - \frac{C_{min}}{C_{max}}\right] = \frac{2270}{1000}\left[1 - \frac{1000}{2500}\right]$$
$$= 1.364 \text{ (dimensionless)}$$

and the effectiveness of the exchanger is [equation (5)]

$$\varepsilon = \frac{1 - e^{-(NTU)B}}{1 - D\,e^{-(NTU)B}}, \quad D = \frac{m_e}{m_o} = \frac{1}{C_{max}/C_{min}} = \frac{1000}{2500}$$

$$\varepsilon = \frac{1 - e^{-1.364}}{1 - (0.4)e^{-1.364}} = 83\%.$$

The outlet temperatures may now be calculated from the specified inlet conditions by means of the definition of the overall exchanger effectiveness:

Data:

Air capacity rate,	$C_a = 2500$ B.t.u./(hr °F)
Hot material capacity rate,	$C_h = 1000$ B.t.u./(hr °F)
Chain capacity rate at an exchanger rotation rate of 40 r/hr, modified to account for wall effects,	$C_c = 10{,}000$ B.t.u./(hr °F)
Temperatures	
Air inlet,	$t_{a1} = 100$°F
Hot material inlet,	$t_{h_2} = 1980$°F
Heat transfer area	
Total chain area, modified to account for wall effects,	$= 770$ ft^2
Distributed as	
Chain to hot material,	$A_{ch} = 297$ ft^2
Chain to air,	$A_{ca} = 473$ ft^2
Conductance data	
Hot material to chains,	$U_{ch} = 20$ B.t.u./(hr ft^2 °F)
Chains to air,	$U_{ca} = 8$ B.t.u./(hr ft^2 °F).

Calculation of the NTUs for the transfers from hot material to chains and from chains to air and substitution into equations (9a) and (10a) will yield the effectivenesses ε_{ch} and ε_{ca}.

$$NTU_{ch} = \frac{(20)(297)}{10{,}000} = 0.595 \text{ (dimensionless)}$$

$$NTU_{ca} = \frac{(8)(473)}{10{,}000} = 0.387 \text{ (dimensionless)}$$

$$\varepsilon_{ch} = 1 - e^{-NTU_{ch}} = 1 - e^{-0.595} = 0.448$$

$$\varepsilon_{ca} = 1 - e^{-NTU_{ca}} = 1 - e^{-0.378} = 0.315.$$

These effectivenesses, combined with the chain capacity rate, yield the effective overall conductance–area product [equation (16a)].

$$(AU)_0 = \left[\frac{1}{\frac{1}{\varepsilon_{ch}} + \frac{1}{\varepsilon_{ca}} - 1}\right] C_c$$

$$= \left[\frac{1}{\frac{1}{0.448} + \frac{1}{0.315} - 1}\right] 10{,}000$$

$$(AU)_0 = 2270 \text{ B.t.u./(hr °F)}.$$

$$\varepsilon = \frac{t_{h_2} - t_{h_1}}{t_{h_2} - t_{a_1}} = 0.83 = \frac{1980 - t_{h_1}}{1980 - 100}$$

$$t_{h_1} = 420\text{°F}.$$

An energy balance on the exchanger indicates a final air temperature of 723°F.

Basic conductance data for the complex flow system of the exchanger considered are not readily available. This, combined with the difficulty in properly evaluating the transfer area and a chain capacity rate which will account for the effect of the exchanger walls, makes the results of the foregoing analysis of questionable quantitative value. Nevertheless, the results of such an analysis are useful in extrapolating the known performance of a given exchanger to new conditions of operation, such as a change in the rotative speed and changes in air and hot material flow rates.

Suppose the experimentally evaluated effectiveness of an exchanger of this type to be 83%. This, together with the known capacity rates (which are the same as those given above), indicates a value of $(NTU)B = 1.364$ [equation (5)].

It is desired to determine the effect of a change in rotative speed, to a new value of 30 rev/hr, on the performance of the unit.

A consideration of $(AU)_0$ as defined by equation (16a) reveals that as a first approximation $(AU)_0$ varies directly as the chain capacity rate. For a given system, C_c is directly proportional to the rotative speed.

Let the values of $(NTU)B$ and the other quantities associated with the new operating condition be designated by an asterisk *.

$$(NTU)B^* = (NTU)B30/40$$
$$= (1.364)30/40 = 1.023.$$

Substitution of this new magnitude, $(NTU)B^*$, into equation (5) yields the new effectiveness $\varepsilon^* = 75\%$, defining new outlet conditions as follows:

Hot material exit temperature,

$$t_{h_1} = 570°\text{F instead of } 420°\text{F}$$

Air exit temperature,

$$t_{a_2} = 660°\text{F instead of } 723°\text{F}.$$

III. CONCLUSIONS

1. A generalization of the design procedure employed for heat exchangers reveals the convenient extension of the method to problems involving other types of exchangers.

2. Certain exchanger calculations are more readily accomplished by utilization of the effectiveness concept.

3. Use of effectiveness equations as the basis of the analysis for coupled exchanger systems eliminates many of the algebraic intricacies normally encountered.

4. Use of the log-mean potential and the derived effectiveness equations for exchanger systems in which the equilibrium and operating lines are not approximately straight will produce significant errors.

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